UK SENIOR MATHEMATICAL CHALLENGE

Tuesday 7 November 2017

For reasons of space, these solutions are necessarily brief.
There are more in-depth, extended solutions available on the UKMT website, which include some exercises for further investigation:
http://www.ukmt.org.uk

1. **C** Of the options given, 2015 is a multiple of 5, 2016 and 2018 are even, and 2019 is a multiple of 3. So the prime number must be 2017.

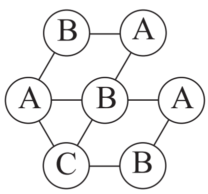
2. **B** Dave's mass per unit length in g/cm is $26 \div 40 = 6.5 \div 10 = 0.65$.

3. **E** In the new arrangement, the sum of the first two numbers and the sum of the last two numbers must be equal. Considering all ten possible pairings, the only two pairs with the same total are 2, 9 and 5, 6. So 14 must be the middle number.

4. **E** By making a common denominator, $2017 - \dfrac{1}{2017} = \dfrac{2017^2 - 1}{2017}$. Then, using the difference of two squares on the numerator, this can be written as $\dfrac{2018 \times 2016}{2017}$.

5. **C** The calculation 13.4 thousand million multiplied by 6×10^{12} gives $13.4 \times 10^9 \times 6 \times 10^{12}$ which is roughly 80×10^{21}. So the distance is roughly 8×10^{22} miles.

6. **B** In the bottom left of the diagram there are three circles connected to each other. These three must be coloured using different colours. Once those have been determined, one possible colouring, using just three colours, is as shown.



7. **C** Simplifying $\sqrt{2} + \sqrt{8} + \sqrt{18}$ gives $\sqrt{2} + 2\sqrt{2} + 3\sqrt{2} = 6\sqrt{2}$. This is the same as $\sqrt{36 \times 2}$ so $\sqrt{72} = \sqrt{k}$ and therefore $k = 72$.

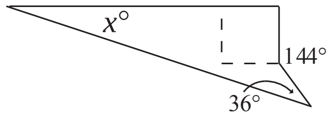
8. **B** Evaluating each option in turn gives $1^{-1} = 1$; $4^{-\frac{1}{2}} = \left(\frac{1}{4}\right)^{\frac{1}{2}} = \frac{1}{2}$; $6^0 = 1$; $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = 2^2 = 4$; $16^{\frac{3}{4}} = \left(16^{\frac{1}{4}}\right)^3 = 2^3 = 8$. Only option B is not an integer.

9. **B** The sum of the areas of the tiles is $(1 \times 2) + (2 \times 3) + (3 \times 4) + \ldots + (8 \times 9)$ $= 2 + 6 + 12 + 20 + 30 + 42 + 56 + 72 = 240 = 15 \times 16$. So the value of n is 15.

10. **D** Let the sides of each of the smaller rectangles be 1 and a with $a > 1$. Then the sides of the larger rectangle are a and 3 with $3 > a$. As the rectangles are similar, $\frac{a}{1} = \frac{3}{a}$. So $a^2 = 3$ and $a = \sqrt{3}$. The ratio of the sides is therefore $\sqrt{3} : 1$.

11. **D** Let the ages of the teenagers be a and b with $a > b$. Then $4(a + b) = a^2 - b^2 = (a + b)(a - b)$ and so $4 = a - b$ as $a + b \neq 0$. Also, $a + b = 8(a - b) = 8 \times 4 = 32$. We now have $a - b = 4$ and $a + b = 32$ so $a = 18$ and $b = 14$. Thus the older of the two is 18.

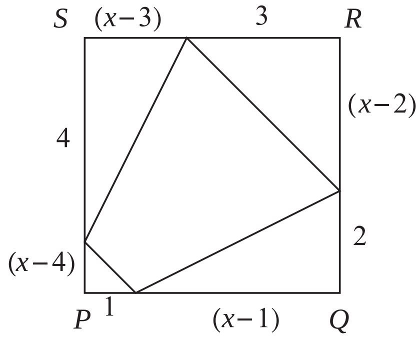
12. **B** Each of the exterior angles of a regular decagon is $360° \div 10 = 36°$ so each interior angle is $180° - 36° = 144°$. The quadrilateral containing x has an angle sum of $360°$. So $x + 90 + (360 - 144) + 36 = 360$ and $x = 18$.



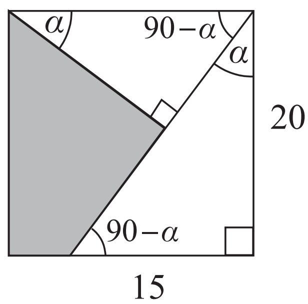
13. **C** Since only one person is lying this must be Genotan or Josh; their statements cannot both be true. If Josh were lying (and therefore guilty) then Genotan's statement would mean that Tegan was also guilty, a contradiction. Hence Genotan is lying and we can check that all four statements are consistent with this.

14. **A** Viewing the diagram as a pentagon inside a hexagon and labelling the unmarked angle in the pentagon as $y°$ gives $4x + y = 540$… (1). For the hexagon, the exterior reflex angle is $y°$ as opposite angles are equal. So the unmarked interior angle is $(360 - y)°$. This gives $5x + 360 - y = 720$, so $5x - y = 360$… (2). Adding equations (1) and (2) gives $9x = 900$ and therefore $x = 100$.

15. **B** Let the length of PQ be x. The sum of the areas of the four right-angled triangles is half the area of the square so $\frac{1}{2}[1 \times (x - 4)] + \frac{1}{2}[4 \times (x - 3)] + \frac{1}{2}[3 \times (x - 2)] + \frac{1}{2}[2 \times (x - 1)] = \frac{1}{2}x^2$. This gives $(x - 4) + 4(x - 3) + 3(x - 2) + 2(x - 1) = x^2$ which simplifies to $10x - 24 = x^2$. Factorising $x^2 - 10x + 24 = 0$ gives $(x - 6)(x - 4) = 0$ so $x = 4$ or 6. For there to be four triangles as shown, $(x - 4) > 0$, so x, the length of PQ, is 6.



16. **D** The area of the shaded quadrilateral is the area of the square less the combined areas of the two right-angled triangles. Using both 'the angle sum of a triangle is 180°' and 'the sum of the angles in the corner of a square is 90°', those two triangles have the same sized angles so they are similar. By Pythagoras' Theorem, the hypotenuse of the larger triangle has length 25. The lengths of the sides of the smaller triangle are then $\frac{20}{25}$ of the lengths of the sides of the larger triangle. So the shaded area is $20^2 - \frac{1}{2} \times 15 \times 20 - \frac{1}{2} \times 12 \times 16$ which is $400 - 150 - 96 = 154$.



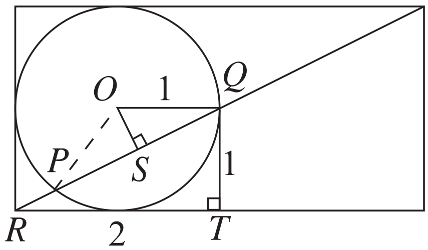
17. **D** Amy's number of sweets is a multiple of 3, so let her start with $3a$ sweets. Once Amy has passed on a of them, Beth's number of sweets must be a multiple of 3. Likewise, after Beth passes on sweets to Claire, Claire's number of sweets must become a multiple of 3. Since $40 = 3 \times 13 + 1$, Beth must pass on $3b - 1$ sweets for some b. So Beth must have had $9b - 3$ sweets after receiving the a sweets from Amy and hence $9b - 3 - a$ to start with. This allows us to fill in the table:

	Amy	Beth	Claire
originally	$3a$	$9b - 3 - a$	40
after Amy's gift to Beth	$2a$	$9b - 3$	40
after Beth's gift to Claire	$2a$	$6b - 2$	$3b + 39$
after Claire's gift to Amy	$2a + b + 13$	$6b - 2$	$2b + 26$

As the girls end up with the same number of sweets, $6b - 2 = 2b + 26$ and so $b = 7$. As $6 \times 7 - 2$ and $2 \times 7 + 26$ both equal 40, the girls each have 40 sweets at the end. Hence Amy's finishing number $2a + 7 + 13 = 40$ so $a = 10$. Then Beth's original number is $9 \times 7 - 10 - 3 = 50$.

18. **E** Given the ratio $A : G$ is $5 : 4$, we have $4A = 5G$, so $4 \times \frac{1}{2}(x + y) = 5\sqrt{xy}$. Squaring gives $4(x + y)^2 = 25xy$ and hence $4x^2 + 8xy + 4y^2 = 25xy$. So $4x^2 - 17xy + 4y^2 = 0$ which factorises to give $(4x - y)(x - 4y) = 0$ so that either $4x = y$ or $x = 4y$. However the first case is impossible as we are given $x > y > 0$. Hence $x = 4y$ and $x : y$ is $4 : 1$.

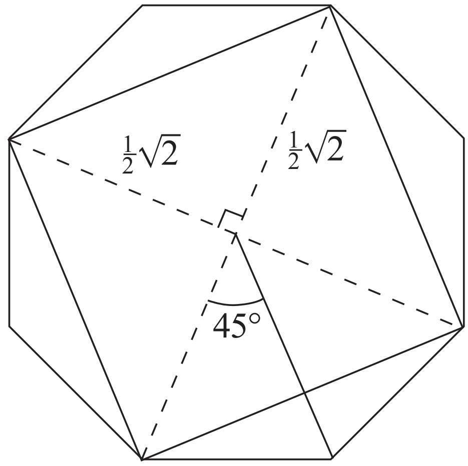
19. **B** Let O be the centre of the circle, R be a corner of the rectangle, T be the point on the rectangle directly below Q and S be the midpoint of PQ. In triangle QRT we have $QT = 1$ and $RT = 2$ and therefore $RQ = \sqrt{5}$, by Pythagoras' Theorem. As RT and OQ are parallel, the right-angled triangles OQS and QRT are similar, so $QS = \frac{2}{\sqrt{5}}$. Therefore the length of chord PQ is $2 \times \frac{2}{\sqrt{5}} = \frac{4}{\sqrt{5}}$.



20. **B** The length of the radius of the quarter circle TRU is the length of PR, which by Pythagoras' Theorem is $\sqrt{2}$. The lengths PU and PR are equal as they are radii of the same circle. So $SU = PU - PS = \sqrt{2} - 1$. Hence the total length of the four quarter circles making up the perimeter of the shaded region is $2 \times \frac{1}{4} \times 2\pi\sqrt{2} + 2 \times \frac{1}{4} \times 2\pi(\sqrt{2} - 1) = (2\sqrt{2} - 1)\pi$.

21. **C** The equation $4^x = y^2 + 15$ can be rearranged to give $(2^x)^2 - y^2 = 15$ which, using the difference of two squares, is $(2^x + y)(2^x - y) = 15$. As we are looking for positive integer values then either $2^x + y = 5$ and $2^x - y = 3$ or $2^x + y = 15$ and $2^x - y = 1$. Solving each pair gives either $(x, y) = (2, 1)$ or $(x, y) = (3, 7)$ so there are just two possible pairs.

22. **D** The square can be split into four congruent right-angled triangles by joining each of its vertices to the centre. Each edge of the square has length 1, so the shorter sides of each triangle have length $\frac{1}{2}\sqrt{2}$.



The regular octagon can be split into eight congruent isosceles triangles whose longer sides have length $\frac{1}{2}\sqrt{2}$, by joining all eight vertices to the centre. These sides are separated by an angle of $\frac{1}{8} \times 360° = 45°$. Using the formula 'area $= \frac{1}{2}ab \sin C$', the total area of the octagon is $8 \times \frac{1}{2} \times \frac{1}{2}\sqrt{2} \times \frac{1}{2}\sqrt{2} \sin 45° = 2 \sin 45° = 2 \times \frac{1}{\sqrt{2}} = \sqrt{2}$.

23. **C** Reflecting a graph in the line $y = x + 2$ is algebraically equivalent to replacing y by $x + 2$ and $x + 2$ by y in the equation of the graph. So $y = x^2$ becomes $(x + 2) = (y - 2)^2$. That is $x + 2 = y^2 - 4y + 4$ which simplifies to $x = y^2 - 4y + 2$.

24. **D** When n non-parallel lines are drawn in a plane, each line intersects every other line exactly once. So each line intersects $(n - 1)$ lines. A line drawn parallel to one of the existing n lines also intersects $(n - 1)$ lines. In order that every line drawn intersects the same number of other lines, each of the original n lines must be part of a set of k parallel lines, for some integer k. Then kn lines will each have $k(n - 1)$ points of intersection. Here $k(n - 1) = 10$ so as $(n - 1)$ is an integer $(n - 1)$ must be 1, 2, 5 or 10 giving $n = 2, 3, 6$ or 11 and $k = 10, 5, 2$ and 1 respectively. Of the options given: A is 11 non-parallel lines ($n = 11, k = 1$); B is 6 sets of 2 parallel lines ($n = 6, k = 2$); C is 3 sets of 5 parallel lines ($n = 3, k = 5$); E is 2 different sets of 10 parallel lines ($n = 2, k = 10$). These are the only four possible solutions to the problem, so 16 could not be the number of lines.

25. **A** Let the side-length of S be 1 and the side-length of N be x. The fraction of the area of N that is the area of S is then $\left(\frac{1}{x}\right)^2 = \frac{1}{x^2}$. Let the length of the hypotenuse of each of the nine triangles be $1 + h$, then the third side of each triangle is h. Using Pythagoras' Theorem, $h^2 + x^2 = (1 + h)^2$ so $h^2 + x^2 = 1 + 2h + h^2$ and the fraction we want, $\frac{1}{x^2}$, is then $\frac{1}{1 + 2h}$. The angle in each triangle which is adjacent to S is also an exterior angle of S, so is $\frac{1}{9} \times 360° = 40°$. We have $\cos 40° = \dfrac{h}{1 + h}$ which rearranges to give $h = \dfrac{\cos 40°}{1 - \cos 40°}$. Therefore $1 + 2h = \left(\dfrac{1 - \cos 40°}{1 - \cos 40°}\right) + \dfrac{2 \cos 40°}{1 - \cos 40°} = \dfrac{1 + \cos 40°}{1 - \cos 40°}$. So the fraction of the area N which is the area of S is $\dfrac{1}{1 + 2h} = \dfrac{1 - \cos 40°}{1 + \cos 40°}$.